Int J Theor Phys (2008) 47: 280–290
DOI 10.1007/s10773-007-9472-5

Tensor Product of Distributive Sequential Effect Algebras and Product Effect Algebras

Eissa D. Habil

Received: 28 August 2006 / Accepted: 8 June 2007 / Published online: 8 November 2007

Abstract A distributive sequential effect algebra (DSEA) is an effect algebra on which a distributive sequential product with natural properties is defined. We define the tensor product of two arbitrary DSEA's and we give a necessary and sufficient condition for it to exist. As a corollary we obtain the result (see Gudder, S. in Math. Slovaca 54:1–11, 2004, to appear) that the tensor product of a pair of commutative sequential effect algebras exists if and only if they admit a bimorphism. We further obtain a similar result for the tensor product of a pair of product effect algebras.

Keywords Effect algebras · Sequential products · Distributive sequential products · Tensor products · Product effect algebras · Fuzzy sets

1 Introduction

Sequential effect algebras (abbreviated: SEA's) have been recently introduced by Gudder and Greechie in [2, 3] to study general properties of sequential measurements [2–4]. Important physical models for SEA's can be constructed from fuzzy set systems and Hilbert space operators [3, 5–8]. In [1], Gudder studied tensor products of SEA's because they describe combined physical systems. Gudder proved the existence of tensor product of pairs of SEA's that are commutative, but not for arbitrary pairs of SEA's.

In this paper, we introduce a stronger and a natural definition of a sequential product on an effect algebra, which we call a distributive sequential product and thus introduce what we shall call distributive sequential effect algebras (abbreviated: DSEA's). This new class properly contains the class of commutative SEA's and is properly contained in the class of SEA's. Then we give a necessary and sufficient condition for the existence of the tensor product of two arbitrary DSEA's. As a corollary we obtain the result (see [1]) that the tensor product of a pair of commutative SEA's exists if and only if they admit a bimorphism.

E.D. Habil (✉)
Department of Mathematics, Islamic University of Gaza, P.O. Box 108, Gaza, Palestine
e-mail: habil@iugaza.edu.ps

Another related class of effect algebras called product effect algebras (abbreviated: PEA's) has been introduced and studied by Dvurečenskij in [9]. As a second corollary to our main result, we obtain the result that the tensor product of two PEA's exists if and only if they admit a bimorphism. Our presentation parallels the beautiful accounts of [1] and [10].

Throughout this paper, the symbol **R** denotes the set of all real numbers, and the notation $:=$ means "equals by definition".

2 Basic Definitions

In this section we summarize the basic definitions concerning effect algebras [11] and sequential effect algebras [2, 3], and we present the definition of a distributive sequential effect algebra. An *effect algebra* is a system $(E, 0, 1, \oplus)$ consisting of a set E containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions $\forall a, b, c \in E$:

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(EA2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(EA3) For every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.
(EA4) If $1 \oplus a$ is defined, then $a = 0$.

We shall write E for the effect algebra $(E, 0, 1, \oplus)$ if there is no danger of misunderstanding. Let E be an effect algebra and $a, b \in E$. Following [11], we say that a is *orthogonal* to b in E and write $a \perp b$ if and only if $a \oplus b$ is defined in L. We define $a \leq b$ to mean that there exists $c \in E$ such that $a \perp c$ and $b = a \oplus c$. The unique element $b \in E$ corresponding to a in Condition (EA3) above is called the *orthosupplement* of a and is written as $a' := b$. For any effect algebra E, it can be easily proved (see [11]) that $0 \leq a \leq 1$ holds for all $a \in E$, that $a \perp b$ iff $a \leq b'$, that, with $\leq$ as defined above, $(E, \leq, 0, 1)$ is a partially ordered set.

Example 2.1 Consider the unit interval $[0, 1] \subseteq \mathbf{R}$, and for $a, b \in [0, 1]$, define $a \perp b$ if $a + b \leq 1$ in which case $a \oplus b := a + b$. It is easy to check that $([0, 1], 0, 1, \oplus)$ is an effect algebra.

The following example plays an important role for unsharp measurements of quantum mechanics [5, 11].

Example 2.2 Consider the set $\mathcal{E}(H)$ of all self-adjoint operators A on a Hilbert space H with $O \leq A \leq I$, where O and I are the zero and the identity operators, respectively, on H. For $A, B \in \mathcal{E}(H)$, define

$$A \oplus B := A + B \quad \text{iff} \quad A + B \leq I.$$

It is not difficult to show that, under this $\oplus$, the system $(\mathcal{E}(H), O, I, \oplus)$ forms an effect algebra [11].

Let E and F be effect algebras. A mapping $\phi : E \to F$ is called a *morphism* iff $\phi(1) = 1$ and for $a, b \in E$, $a \perp b \Rightarrow \phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. A mapping $\phi : E \to F$ is called *a monomorphism* if ϕ is a morphism and if $\phi(a) \perp \phi(b) \Rightarrow a \perp b$. A surjective monomorphism is an *isomorphism*. One can easily check that a morphism $\phi : E \to F$ is an isomorphism iff ϕ is bijective and ϕ^{-1} is a morphism. For more about morphisms of effect algebras (which are essentially the same as morphisms of orthoalgebras), we refer the reader to [12, 13].

Let E, F, G be effect algebras. A mapping $\beta : E \times F \to G$ is called a bimorphism if and only if

(B1) $\beta(1, 1) = 1$;
(B2) $\beta(a \oplus b, c) = \beta(a, c) \oplus \beta(b, c)$ whenever $a, b \in E$ with $a \perp b$;
(B3) $\beta(a, b \oplus c) = \beta(a, b) \oplus \beta(a, c)$ whenever $b, c \in E$ with $b \perp c$.

If $\beta : E \times F \to G$ is a bimorphism, then it is easy to check that $\beta(\cdot, 1)$ and $\beta(1, \cdot)$ are morphisms.

Example 2.3 Let H_1 and H_2 be two Hilbert spaces over the same field and let $\mathcal{E}(H_1) \otimes \mathcal{E}(H_2)$ be the standard Hilbert space tensor product and define a mapping $\beta : \mathcal{E}(H_1) \times \mathcal{E}(H_2) \to \mathcal{E}(H_1) \otimes \mathcal{E}(H_2)$ by $\beta(A_1, A_2) := A_1 \otimes A_2$. Then it is easy to check that β is an effect algebra bimorphism [10, 12].

For a binary operation $\circ$, if $a \circ b = b \circ a$ we write $a \mid b$. A *sequential effect algebra* (abbreviated: SEA)[2] is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ : E \times E \to E$ is a binary operation that satisfies the following conditions.

(S1) $b \longmapsto a \circ b$ is additive for every $a \in E$.
(S2) $1 \circ a = a$ for all $a \in E$.
(S3) If $a \circ b = 0$, then $a \mid b$.
(S4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in E$.
(S5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$ whenever $a \oplus b$ is defined.

A binary operation $\circ$ on E that satisfies (S1)–(S5) is called a *sequential product* on E. If $a \mid b$ for all $a, b \in E$, then E is called a *commutative* SEA.

The effect algebra [0, 1] of Example 2.1 becomes a commutative SEA when $a \circ b := ab$. It has been shown in [2, 8] that the effect algebra $\mathcal{E}(H)$ of Example 2.2 is a (noncommutative) SEA under the operation $A \circ B := A^{\frac{1}{2}} B A^{\frac{1}{2}}$ where $A^{\frac{1}{2}}$ is the unique positive square root of A. This Hilbert space SEA is useful for studying the foundations of quantum mechanics [4–8]. Here are more examples of interesting SEA's.

Example 2.4 Every Boolean algebra $(B, 0, 1, \oplus, \circ)$ is a commutative SEA under the operations $a \oplus b := a \vee b$ whenever $a \wedge b = 0$, and $a \circ b := a \wedge b$ for all $a, b \in B$.

Example 2.5 Let X be a nonempty set and let

$$[0, 1]^X := \{f : f \text{ is a function from } X \text{ to } [0, 1]\}.$$

Let $f_0, f_1 \in [0, 1]^X$ be defined by $f_0(x) := 0$, $f_1(x) := 1$ for all $x \in X$. A subset $\mathcal{F} \subseteq [0, 1]^X$ is called a *fuzzy set system* on X [14] if $f_0, f_1 \in \mathcal{F}$, if $f \in \mathcal{F}$ then $f_1 - f \in \mathcal{F}$, if $f, g \in \mathcal{F}$ with $f + g \leq 1$ then $f + g \in \mathcal{F}$ and if $f, g \in \mathcal{F}$ then $fg \in \mathcal{F}$. Then a fuzzy set system $\mathcal{F}$ becomes a commutative SEA when $f \oplus g := f + g$ for $f + g \leq 1$ and $f \circ g := fg$.

For more about SEA's and their properties, we refer the reader to [2, 3]. The following result has been proven in [2, 8].

Theorem 2.6 *For $A, B \in \mathcal{E}(H)$ we have $B = A \circ B \oplus A' \circ B$ if and only if $A \mid B$ if and only if $AB = BA$.*

Remark 2.7 By (S1), the sequential product $\circ$ on any SEA E is always distributive on the right; that is,

$$c \circ (a \oplus b) = c \circ a \oplus c \circ b$$

whenever $a \oplus b$ is defined in E. However, it is not always distributive on the left; that is,

$$(a \oplus b) \circ c \neq a \circ c \oplus b \circ c$$

in general. To see this, consider the Hilbert space SEA $\mathcal{E}(H)$. Choose $A, C \in \mathcal{E}(H)$ such that $AC \neq CA$. Then by Theorem 2.6 we have

$$A \circ C \oplus A' \circ C \neq C = I \circ C = (A \oplus A') \circ C.$$

We now strengthen the definition of a SEA E by requiring that the sequential product operation on E be distributive from both left and right in order to delineate a class of SEA's that lies strictly between the class of commutative SEA's and the class of SEA's, which admit tensor products. A *distributive sequential effect algebra* (abbreviated: DSEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ : E \times E \to E$ is a binary operation that satisfies the following conditions.

(DS1) $b \longmapsto a \circ b$ is additive for every $a \in E$.
(DS2) $b \longmapsto b \circ a$ is additive for every $a \in E$.
(DS3) $1 \circ a = a \circ 1 = a$ for all $a \in E$.
(DS4) If $a \circ b = 0$, then $b \circ a = 0$.
(DS5) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in E$.

A binary operation $\circ$ on E that satisfies (DS1)–(DS5) is called a *distributive sequential product* on E.

Lemma 2.8 *Every DSEA is a SEA.*

Proof Let E be a DSEA. We show that E satisfies (S1)–(S5). Clearly, (S1) follows from (DS1), (S2) follows from (DS3), and (S3) follows from (DS4). To show (S4) holds, suppose that $a \mid b$. Then we have

$$a = a \circ 1 = a \circ (b \oplus b') = a \circ b \oplus a \circ b'$$

and

$$a = 1 \circ a = (b \oplus b') \circ a = b \circ a \oplus b' \circ a;$$

so that

$$a \circ b \oplus a \circ b' = b \circ a \oplus b' \circ a.$$

Since $a \circ b = b \circ a$, the cancellation law implies that $a \circ b' = b' \circ a$. Hence $a \mid b'$. The second part of (S4) follows from (DS5). Thus (S4) holds. To verify (S5), suppose that $c \mid a$ and $c \mid b$. Then, by (DS5), we have

$$c \circ (a \circ b) = (c \circ a) \circ b = (a \circ c) \circ b = a \circ (c \circ b) = a \circ (b \circ c) = (a \circ b) \circ c;$$

so that $c \mid a \circ b$. Moreover, if $a \oplus b$ is defined, then, by (DS1) and (DS2), we have

$$c \circ (a \oplus b) = c \circ a \oplus c \circ b = a \circ c \oplus b \circ c = (a \oplus b) \circ c;$$

so that $c \mid (a \oplus b)$. Thus (S5) holds, and therefore E is a SEA. □

Remark 2.9 (1). The converse of Lemma 2.8 is not true; that is, not every SEA is a DSEA. For example, the Hilbert space SEA $\mathcal{E}(H)$ is not a DSEA.

(2). It can be easily checked that every commutative SEA is a DSEA, and therefore our earlier examples of commutative SEA's, namely the unit interval [0, 1], every Boolean algebra, and any fuzzy set system $\mathcal{F} \subseteq [0,1]^X$, are now examples of DSEA's. However, not every DSEA is a commutative SEA, as the following example shows.

Example 2.10 Let $E = HS(I_1, I_2)$ be the horizontal sum of the SEA $(I_1, 0, 1, \oplus_1, \circ_1)$ and the SEA $(I_2, 0, 1, \oplus_2, \circ_2)$, where $I_1 = I_2 = [0, 1]$ (see [2]). Define an orthosum $\oplus$ on E as follows. For $a, b \in E$, define $a \oplus b := a \oplus_i b$ if and only if $a, b \in I_i$ with $a \oplus_i b$ is defined in I_i and no other orthosums are defined on E. Let $\phi_{ij} : I_i \to I_j$, $i \neq j \in \{1, 2\}$ be the natural identification (identity) mapping, and define $\circ$ on E by

$$a \circ b := \begin{cases} a \circ_i b, & \text{if } a, b \in I_i \text{ for some } i \in \{1,2\}, \\ a \circ_i \phi_{ji}(b), & \text{if } a \in I_i \setminus \{0,1\},\ b \in I_j \setminus \{0,1\},\ i \neq j \in \{1,2\}. \end{cases}$$

It has been shown in [2] that under the above-defined operations $\oplus$ and $\circ$, E is a noncommutative SEA.

We now show that E is a DSEA. Note first that (DS1) follows from (S1). To verify (DS2), suppose that $a \oplus b$ is defined in $E, c \in E$. If $a, b, c \in I_i$ for some $i \in \{1, 2\}$, then

$$(a \oplus b) \circ c = (a \oplus_i b) \circ_i c = a \circ_i c \oplus_i b \circ_i c = a \circ c \oplus b \circ c.$$

Otherwise, $a, b \in I_i$, $c \in I_j$, $i \neq j \in \{1, 2\}$, and we have

$$(a \oplus b) \circ c = (a \oplus_i b) \circ_i \phi_{ji}(c) = a \circ_i \phi_{ji}(c) \oplus_i b \circ_i \phi_{ji}(c) = a \circ c \oplus b \circ c.$$

Thus, (DS2) holds. Since E is a SEA, it follows from [14, Lemma 3.1] that $a \circ 1 = 1 \circ a = a$ for all $a \in E$. Thus (DS3) holds. It is clear that (DS4) follows from (S3). To verify (DS5), let $a, b, c \in E$. If $a, b, c \in I_i$ for some $i \in \{1, 2\}$, then

$$a \circ (b \circ c) = a \circ_i (b \circ_i c) = (a \circ_i b) \circ_i c = (a \circ b) \circ c.$$

Otherwise, $a, b \in I_i$, $c \in I_j$, $i \neq j \in \{1, 2\}$, and we have

$$(a \circ b) \circ c = (a \circ_i b) \circ_i \phi_{ji}(c) = a \circ_i (b \circ_i \phi_{ji}(c)) = a \circ (b \circ c).$$

It follows that the class of DSEA's properly contains the class of commutative SEA's and is properly contained in the class of SEA's. It has been proven in [1] that the tensor product of two commutative SEA's exists if and only if they admit a SEA-bimorphism. However, it is not known whether the tensor product of arbitrary SEA's exists [1, 14]. For more about open problems on SEA's we refer the reader to [14]. We shall show in the next section that the tensor product of two arbitrary DSEA's exists if and only if they admit a DSEA-bimorphism.

3 Tensor Products

Let E and F be DSEA's. A DSEA-*morphism* $\phi : E \to F$ is an effect algebra morphism that satisfies

$$\phi(a \circ b) = \phi(a) \circ \phi(b) \quad \text{for every } a, b \in E.$$

A DSEA-morphism that is an effect algebra isomorphism is a DSEA-*isomorphism*. If G is also a DSEA, a DSEA-*bimorphism* is a map $\beta : E \times F \to G$ that is an effect algebra bimorphism satisfying

$$\beta(a, b) \circ \beta(c, d) = \beta(a \circ c, b \circ d)$$

for every $a, c \in E$, $b, d \in F$. The DSEA *tensor product* of E and F is a pair (T, τ) consisting of a DSEA T and DSEA-bimorphism $\tau : E \times F \to T$ such that

(T1) Every $a \in T$ has the form $a = \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$.
(T2) If $\beta : E \times F \to G$ is a DSEA-bimorphism, then there exists a DSEA-morphism $\phi : T \to G$ such that $\beta = \phi \circ \tau$.

It is easy to show that if the DSEA tensor product of two DSEA's exists, then it is unique up to a DSEA-isomorphism; that is, if (T, τ) and (T^*, τ^*) are DSEA tensor products of the DSEA's E and F, then there exists a unique DSEA-isomorphism $\phi : T \to T^*$ such that $\phi(\tau(a, b)) = \tau^*(a, b)$ for all $a \in E, b \in F$.

We now present the main result, which gives a necessary and sufficient condition for pairs E and F of DSEA's to admit a tensor product. A finite sequence $A = \{(a_i, b_i)\}_{i=1}^n$ in $E \times F$ is *orthosummable* if

$$\bigoplus \beta(A) := \bigoplus_{i=1}^{n} \beta(a_i, b_i)$$

is defined for every DSEA-bimorphism β.

Theorem 3.1 *The DSEA tensor product of two DSEA's E and F exists if and only if E and F admit a DSEA-bimorphism $\beta : E \times F \to G$ for some DSEA G.*

Proof The necessity of the condition is evident. For sufficiency, suppose that E and F satisfy the stated condition. Let $\mathcal{K}$ be the set of all finite sequences K in $E \times F$ such that $\bigoplus \beta(K) = 1$ for every DSEA-bimorphism β on $E \times F$. Evidently, $\mathcal{K}$ is nonempty since $\{(1, 1)\} \in \mathcal{K}$. Let $\mathcal{E}(\mathcal{K})$ be the set of all finite sequences $\{(a_i, b_i)\}_{i=1}^n$ in $E \times F$ for which there exists a finite sequence $\{(c_j, d_j)\}_{j=1}^m$ in $E \times F$ such that

$$\{(a_1, b_1), \ldots, (a_n, b_n), (c_1, d_1), \ldots, (c_m, d_m)\} \in \mathcal{K}.$$

Define a relation $\sim$ on $\mathcal{E}(\mathcal{K})$ by $A \sim B$ if and only $\bigoplus \beta(A) = \bigoplus \beta(B)$ for every DSEA-bimorphism β on $E \times F$. Then $\sim$ is an equivalence relation, and for $A \in \mathcal{E}(\mathcal{K})$ we let

$$\pi(A) := \{B \in \mathcal{E}(\mathcal{K}) : B \sim A\},$$

and we let

$$\Pi(\mathcal{K}) := \{\pi(A) : A \in \mathcal{E}(\mathcal{K})\}.$$

Following the construction of [1, 10], we now organize $\Pi(\mathcal{K})$ into a DSEA as follows. Define $0 := \pi(\{(0,0)\}) \in \Pi(\mathcal{K})$ and $1 := \pi(\{(1,1)\}) \in \Pi(\mathcal{K})$. For $A = \{(a_i, b_i)\}_{i=1}^n \in \mathcal{E}(\mathcal{K})$ and $B = \{(c_j, d_j)\}_{j=1}^m \in \mathcal{E}(\mathcal{K})$, we define $\pi(A) \perp \pi(B)$ if and only if

$$C := \{(a_1, b_1), \dots, (a_n, b_n), (c_1, d_1), \dots, (c_m, d_m)\} \in \mathcal{E}(\mathcal{K}). \tag{1}$$

Then, by definition of $\mathcal{E}(\mathcal{K})$, there exists a $D \in \mathcal{E}(\mathcal{K})$ such that

$$\bigoplus \beta(A) \oplus \bigoplus \beta(B) \oplus \bigoplus \beta(D) = 1 \tag{2}$$

for every DSEA-bimorphism β on $E \times F$. We note that the relation $\perp$ is well defined, since if $A' \sim A$ and $B' \sim B$, then for every DSEA-bimorphism β on $E \times F$ we have, by (2), that

$$\bigoplus \beta(A') \oplus \bigoplus \beta(B') \oplus \bigoplus \beta(D) = 1. \tag{3}$$

Hence, for $A' = \{(a_i', b_i')\}_{i=1}^n$ and $B' = \{(c_j', d_j')\}_{j=1}^m$ we have, by(3), that

$$C' := \{(a_1', b_1'), \dots, (a_n', b_n'), (c_1', d_1'), \dots, (c_m', d_m')\} \in \mathcal{E}(\mathcal{K}), \tag{4}$$

and therefore $\pi(A') \perp \pi(B')$.

We define an orthosum $\oplus$ on $\Pi(\mathcal{K})$ as follows. If $\pi(A) \perp \pi(B)$ we let

$$\pi(A) \oplus \pi(B) := \pi(C),$$

where $C \in \mathcal{E}(\mathcal{K})$ is given by (1). We note that the operation $\oplus$ is well defined. Indeed, if $A' \sim A$ and $B' \sim B$, where A', B' are as given above, then, by (4), there exists a set $C' \in \mathcal{E}(\mathcal{K})$ such that $\bigoplus \beta(C) = \bigoplus \beta(C')$ for every DSEA-bimorphism β on $E \times F$. Hence, $\pi(C) = \pi(C')$. Now it can be easily checked that the system $(\Pi(\mathcal{K}), 0, 1, \oplus)$ is an effect algebra. Next, we claim that for $A = \{(a_i, b_i)\}_{i=1}^n \in \mathcal{E}(\mathcal{K})$ and $B = \{(c_j, d_j)\}_{j=1}^m \in \mathcal{E}(\mathcal{K})$ there exists a $C \in \mathcal{E}(\mathcal{K})$ such that

$$\bigoplus \beta(C) = \bigoplus \beta(A) \circ \bigoplus \beta(B) = \bigoplus \beta(B) \circ \bigoplus \beta(A) \tag{5}$$

for every DSEA-bimorphism β on $E \times F$. Indeed, let

$$C := \{(a_i \circ c_j, b_i \circ d_j) : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}.$$

Clearly, C is orthosummable and for every DSEA-bimorphism β on $E \times F$ we have, by properties (DS1), (DS2) and (EA2), that

$$\begin{aligned}
\bigoplus \beta(C) &= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \beta(a_i \circ c_j, b_i \circ d_j) \\
&= \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \beta(a_i \circ c_j, b_i \circ d_j) \\
&= \bigoplus_{j=1}^{m} \left[\bigoplus_{i=1}^{n} (\beta(a_i, b_i) \circ \beta(c_j, d_j)) \right]
\end{aligned}$$

$$= \bigoplus_{j=1}^{m}\left[\left(\bigoplus_{i=1}^{n}(\beta(a_i, b_i)) \circ \beta(c_j, d_j)\right)\right]$$
$$= \left(\bigoplus_{i=1}^{n}(\beta(a_i, b_i))\right) \circ \left(\bigoplus_{j=1}^{m}\beta(c_j, d_j)\right)$$
$$= \bigoplus \beta(A) \circ \bigoplus \beta(B).$$

The second equality in (5) is proved similarly. To complete the proof of the above claim, it remains to show that $C \in \mathcal{E}(\mathcal{K})$. Since $B \in \mathcal{E}(\mathcal{K})$, there exists a $B' \in \mathcal{E}(\mathcal{K})$ such that

$$\bigoplus \beta(B) \oplus \bigoplus \beta(B') = 1.$$

Then, by the above argument with B' in role of B, there exists a finite sequence C' in $E \times F$ that is orthosummable and satisfies

$$\bigoplus \beta(C') = \bigoplus \beta(A) \circ \bigoplus \beta(B')$$

for every DSEA-bimorphism β on $E \times F$. Hence, for every DSEA-bimorphism β on $E \times F$ we have

$$\begin{aligned}
\bigoplus \beta(C) \oplus \bigoplus \beta(C') &= \left(\bigoplus \beta(A) \circ \bigoplus \beta(B)\right) \oplus \left(\bigoplus \beta(A) \circ \bigoplus \beta(B')\right) \\
&= \left(\bigoplus \beta(A)\right) \circ \left(\left(\bigoplus \beta(B) \oplus \bigoplus \beta(B')\right)\right) \\
&= \left(\bigoplus \beta(A)\right) \circ 1 \\
&= \bigoplus \beta(A).
\end{aligned}$$

Since $A \in \mathcal{E}(\mathcal{K})$, it follows that $C \in \mathcal{E}(\mathcal{K})$, as desired.

We define a binary operation $\circ$ on $\Pi(\mathcal{K})$ by

$$\pi(A) \circ \pi(B) := \pi(C)$$

where C is given by (5) of the above claim. It follows from the above claim that the operation $\circ$ is well defined. We now show that $\circ$ is a distributive sequential product on $\Pi(\mathcal{K})$. Suppose that $\pi(A), \pi(B), \pi(C) \in \Pi(\mathcal{K})$ are such that $\pi(B) \perp \pi(C)$. Then, by the first part of the above claim applied twice to A, B and A, C, there exist $D, H \in \mathcal{E}(\mathcal{K})$ such that for every DSEA-bimorphism β on $E \times F$ we have

$$\begin{aligned}
\bigoplus \beta(D) &= \bigoplus \beta(A) \circ \bigoplus \beta(B), \\
\bigoplus \beta(H) &= \bigoplus \beta(A) \circ \bigoplus \beta(C).
\end{aligned} \tag{6}$$

Hence, for every DSEA-bimorphism β on $E \times F$ we have, by (6), that

$$\bigoplus \beta(D) \oplus \bigoplus \beta(H) = \bigoplus \beta(A) \circ \left[\bigoplus \beta(B) \oplus \bigoplus \beta(C)\right], \tag{7}$$

which implies that $\bigoplus\beta(D)\perp\bigoplus\beta(H)$. Therefore, it follows from this, (6) and (7) that $\pi(A)\circ\pi(B)\perp\pi(A)\circ\pi(C)$ and

$$\pi(A)\circ[\pi(B)\oplus\pi(C)]=\pi(D)\oplus\pi(H)=\pi(A)\circ\pi(B)\oplus\pi(A)\circ\pi(C)$$

so that (DS1) holds. Using the second part of the above claim (see (5)), it can be shown that a similar argument yields

$$[\pi(B)\oplus\pi(C)]\circ\pi(A)=\pi(B)\circ\pi(A)\oplus\pi(C)\circ\pi(A)$$

so that (DS2) holds. For $K\in\mathcal{K}$ and $A\in\mathcal{E}(\mathcal{K})$ we have $\bigoplus\beta(K)=1$ and so

$$\bigoplus\beta(A)=\bigoplus\beta(K)\circ\bigoplus\beta(A)=\bigoplus\beta(A)\circ\bigoplus\beta(K)$$

for every DSEA-bimorphism β on $E\times F$. Hence,

$$1\circ\pi(A)=\pi(K)\circ\pi(A)=\pi(A)\quad\text{and}\quad\pi(A)\circ 1=\pi(A)\circ\pi(K)=\pi(A)$$

so that (DS3) holds. Suppose that $\pi(A)\circ\pi(B)=0$. Then for every DSEA-bimorphism β on $E\times F$ we have $\bigoplus\beta(A)\circ\bigoplus\beta(B)=0$ so that, by property (DS4), $\bigoplus\beta(B)\circ\bigoplus\beta(A)=0$ and hence $\pi(B)\circ\pi(A)=0$. Thus, (DS4) holds. For every $A,B,C\in\mathcal{E}(\mathcal{K})$ and every DSEA-bimorphism β on $E\times F$ we have

$$\bigoplus\beta(A)\circ\Big[\bigoplus\beta(B)\circ\bigoplus\beta(C)\Big]=\Big[\bigoplus\beta(D)\circ\bigoplus\beta(B)\Big]\circ\bigoplus\beta(C).$$

Hence,

$$\pi(A)\circ[\pi(B)\circ\pi(C)]=[\pi(A)\circ\pi(B)]\circ\pi(C)$$

and (DS5) holds. We conclude that $(\Pi(\mathcal{K}),0,1,\oplus,\circ)$ is a DSEA.

Finally, to construct the desired bimorphism, notice, first, that for every $(a,b)\in E\times F$ we have $\{(a,b)\}\in\mathcal{E}(\mathcal{K})$. Indeed, if $C:=\{(a,b),(a,b'),(a',1)\}$, then for every DSEA-bimorphism β on $E\times F$ we have $\bigoplus\beta(C)=1$ so that $C\in\mathcal{K}$ and therefore $\{(a,b)\}\in\mathcal{E}(\mathcal{K})$. Now define $\tau:E\times F\to\Pi(\mathcal{K})$ by

$$\tau(a,b):=\pi[\{(a,b)\}]$$

for every $(a,b)\in E\times F$. Clearly, $\tau(1,1)=1$. For $a\in E$ and for $b,c\in F$ with $b\perp c$, we have

$$\beta(a,b\oplus c)=\beta(a,b)\oplus\beta(a,c)$$

for every DSEA-bimorphism β on $E\times F$, so that $\{(a,b\oplus c)\}\sim\{(a,b),(a,c)\}$. Hence,

$$\tau(a,b\oplus c)=\tau(a,b)\oplus\tau(a,c)$$

and similarly, for $a,d\in E$ with $a\perp d$ and for $b\in F$, we have

$$\tau(a\oplus d,b)=\tau(a,b)\oplus\tau(d,b).$$

For $(a,b),(c,d)\in E\times F$ we have

$$\beta(a,b)\circ\beta(c,d)=\beta(a\circ c,b\circ d)$$

for every DSEA-bimorphism β on $E \times F$. Hence,

$$\pi[\{(a,b)\}] \circ \pi[\{(c,d)\}] = \pi[\{(a \circ c, b \circ d)\}]$$

and we have

$$\tau(a,b) \circ \tau(c,d) = \tau(a \circ c, b \circ d).$$

Thus, τ is a DSEA-bimorphism. Moreover, any element in $\Pi(\mathcal{K})$ has the form

$$\begin{aligned}\pi[\{(a_1,b_1),\dots,(a_n,b_n)\}] &= \pi[\{(a_1,b_1)\}] \oplus \cdots \oplus \pi[\{(a_n,b_n)\}]\\ &= \tau(a_1,b_1) \oplus \cdots \oplus \tau(a_n,b_n)\end{aligned}$$

for some $\{(a_1,b_1),\dots,(a_n,b_n)\} \in \mathcal{E}(\mathcal{K})$. Finally, to show that τ has the desired universal property, let $\beta : E \times F \to G$ be a DSEA-bimorphism. Define $\phi : \Pi(\mathcal{K}) \to G$ by

$$\phi[\pi(A)] := \bigoplus \beta(A).$$

Evidently, ϕ is well defined and it is easy to check that ϕ is a DSEA-morphism. Moreover,

$$\beta(a,b) = \phi[\pi(\{(a,b)\})] = \phi \circ \tau(a,b)$$

for every $(a,b) \in E \times F$, so that $\beta = \phi \circ \tau$. Thus, we have shown that $(\Pi(\mathcal{K}), \tau)$ is the DSEA tensor product of the DSEA's E and F. □

Since every commutative SEA is a DSEA, we get, as a consequence of our result, the following result which is proved in [1].

Corollary 3.2 *The SEA tensor product of two commutative SEA's exists if and only if they admit a SEA-bimorphism.*

Dvurečenskij [9] has introduced the following definition. A *product effect algebra* (abbreviated: PEA) is a system $(E, 0, 1, \oplus, \cdot)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\cdot : E \times E \to E$ is a binary operation that satisfies the following conditions.

(PE1) $b \longmapsto a \cdot b$ is additive for every $a \in E$.
(PE2) $b \longmapsto b \cdot a$ is additive for every $a \in E$.

A binary operation $\circ$ on E that satisfies (PE1) and (PE2) is called a *product* on E. An element u of a PEA E is said to be a *product unity*, if $a \cdot u = u \cdot a = a$ for every $a \in E$. A product $\cdot$ on E is

(i) *associative* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in E$;
(ii) *commutative* if $a \cdot b = b \cdot a$ for every $a, b \in E$.

It should be noted from the definitions of DSEA's and PEA's that every DSEA $(E, 0, 1, \oplus, \circ)$ is an associative PEA having 1 as a product unity. Thus all our earlier examples of DSEA's are now examples of PEA's. Dvurečenskij has studied in [9] the category of PEA's with the Riesz decomposition property and having 1 as a product unity.

We now study the PEA tensor product of PEA's having 1 as a product unity. We define *PEA-morphisms* and *PEA-bimorphisms* in exactly the same way we defined them for DSEA's. The *PEA tensor product* of two PEA's E and F having 1 as a product unity is a pair (T, τ) consisting of a PEA T having 1 as a product unity and PEA-bimorphism $\tau : E \times F \to T$ such that

(T1) Every $a \in T$ has the form $a = \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$.
(T2) If $\beta : E \times F \to G$ is a DSEA-bimorphism, then there exists a DSEA-morphism $\phi : T \to G$ such that $\beta = \phi \circ \tau$.

By re-examining the proof of Theorem 3.1, we notice that it contains the proof of the following result.

Theorem 3.3 *The PEA tensor product of two PEA's E and F having* 1 *as a product unity exists if and only if E and F admit a PEA-bimorphism $\beta : E \times F \to G$ for some PEA G having* 1 *as a product unity.*

References

1. Gudder, S.: Tensor products of sequential effect algebras. Math. Slovaca **54**, 1–11 (2004)
2. Gudder, S., Greechie, R.: Sequential products on effect algebras. Rep. Math. Phys. **49**, 87–111 (2002)
3. Gudder, S., Greechie, R.: Uniqueness and order in sequential effect algebras. Int. J. Theor. Phys. **44**, 755–770 (2005)
4. Gudder, S.: A historic approach to quantum mechanics. J. Math. Phys. **39**, 5772–5788 (1998)
5. Busch, P., Lahti, P.J., Middlestaedt, P.: The Quantum Theory of Measurements. Springer, Berlin (1991)
6. Busch, P., Grabawski, M., Lahti, P.J.: Operational Quantum Physics. Springer, Berlin (1995)
7. Gudder, S., Nagy, G.: Sequential independent effects. Proc. Am. Math. Soc. **130**, 1125–1130 (2001)
8. Gudder, S., Nagy, G.: Sequential quantum measurements. J. Math. Phys. **42**, 5212–5222 (2001)
9. Dvurečenskij, A.: Product effect algebras. Slovak Academy of Sciences Preprint Series, Preprint 2/2002
10. Dvurečenskij, A.: Tensor product of difference posets. Trans. Am. Math. Soc. **347**, 1043–1057 (1995)
11. Foulis, D.J., Bennett, M.K.: Effect algebras and unsharp quantum logics. Found. Phys. **24**, 1325–1346 (1994)
12. Gudder, S.: Morphisms, tensor products and σ-effect algebras. Rep. Math. Phys. **42**, 321–346 (1998)
13. Habil, E.: Morphisms and pasting of orthoalgebras. Math. Slovaca **47**, 405–416 (1997)
14. Gudder, S.: Open problems for sequential effect algebras. Int. J. Theor. Phys. **44**, 2199–2206 (2005)